

On multiple recurrence.

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1. Introduction.

Let N be a natural number and

$$a_k(N) = \frac{1}{N} \max\{|A| : A \subseteq [1, N],$$

A — does not contain an arithmetic progression of length k \},

where $|A|$ denotes the cardinality of a set A . In [1] P. Erdos and P. Turan realised that it ought to be possible to find arithmetic progression of length k in any set with positive density. In other words they conjectured that for any $k \geq 3$

$$a_k(N) \rightarrow 0, \text{ as } N \rightarrow \infty \quad (1)$$

In case $k = 3$ conjecture (1) was proved by K.F. Roth in [2]. In his paper Roth used the Hardy – Littlewood method to prove the inequality

$$a_3(N) \ll \frac{1}{\log \log N}.$$

At this moment the best result about a lower bound for $a_3(N)$ belongs to J. Bourgain [3]. He proved that

$$a_3(N) \ll \sqrt{\frac{\log \log N}{\log N}}. \quad (2)$$

For an arbitrary k conjecture (1) was proved by E. Szemerédi [4] in 1975.

The second proof of Szemerédi's theorem was given by H. Furstenberg in [10], using ergodic theory. Furstenberg showed that Szemerédi's theorem is equivalent to the multiple recurrence of almost every point in an arbitrary dynamical system. Here we formulate this theorem in the case of metric spaces :

Theorem 1.1 *Let X be a metric space with metric $d(\cdot, \cdot)$ and Borel sigma-algebra of measurable sets Φ . Let T be a measurable map of X into itself preserving the measure μ and let $k \geq 3$. Then*

$$\liminf_{n \rightarrow \infty} \max\{d(T^n x, x), d(T^{2n} x, x), \dots, d(T^{(k-1)n} x, x)\} = 0.$$

for almost all $x \in X$.

Actually, H. Furstenberg obtained more general result.

Theorem 1.2 *Let X be a metric space with metric $d(\cdot, \cdot)$ and Borel sigma-algebra of measurable sets Φ . Let $l \in \mathbf{N}$ and T_1, \dots, T_l be commutative measurable maps of X into itself preserving the measure μ . Then*

$$\liminf_{n \rightarrow \infty} \max\{d(T_1^n x, x), d(T_2^{2n} x, x), \dots, d(T_l^n x, x)\} = 0.$$

for almost all $x \in X$.

Unfortunately, Szemerédi's methods give very weak upper bound for $a_k(N)$. Furstenberg's proof gives no bound. Only in 2001 W.T. Gowers [5] obtained a quantitative result about the speed of tending to zero of $a_k(N)$ with $k \geq 4$. He proved the following theorem.

Theorem 1.3 *Let $\delta > 0$, $k \geq 4$ and $N \geq \exp \exp(C\delta^{-K})$, where $C, K > 0$ is absolute constants. Let $A \subseteq \{1, 2, \dots, N\}$ be a set of cardinality at least δN . Then A contains an arithmetic progression of length k .*

In other words, W.T. Gowers proved that for any $k \geq 4$, we have $a_k(N) \ll 1/(\log \log N)^{c_k}$, where constant c_k depends on k only.

Consider the following problem. Let us consider the two-dimensional lattice $[1, N]^2$ with basis $\{(1, 0), (0, 1)\}$. Define

$$L(N) = \frac{1}{N^2} \max\{|A| : A \subseteq [1, N]^2 \text{ and}$$

A — does not contain any triple $\{(k, m), (k + d, m), (k, m + d), d > 0\}$

with positive $d\}$. (3)

A triple from (3) will be called a "*corner*". In papers [7, 10] shown that $L(N)$ tends to 0 as N tends to infinity. W.T. Gowers (see [5]) set a question about the speed of convergence to 0 of $L(N)$.

In [8] V. Vu proposed the following solution. Let us define $\log_* N$ as the largest integer k such that $\log_{[l]} N \geq 2$, where $\log_{[1]} N = \log N$ and for $l \geq 2$ $\log_{[l]} = \log(\log_{[l-1]} N)$. V. Vu proved that

$$L(N) \leq \frac{100}{\log_*^{1/4} N}$$

The main result of [6] is

Theorem 1.4 *Let $\delta > 0$, $N \geq \exp \exp \exp(\delta^{-c})$, where $c > 0$ is an absolute constant. Let $A \subseteq \{1, \dots, N\}^2$ be a set of cardinality at least δN^2 . Then A contains a triple $(k, m), (k + d, m), (k, m + d)$, where $d > 0$.*

Theorem 1.4 implies that $L(N) \ll 1/(\log \log \log N)^{C_1}$, where $C_1 > 0$ is an absolute constant.

In the present paper we apply Theorem 1.4 to the theory of dynamical systems and obtain result about the multiple recurrence of almost every point in an arbitrary dynamical system with two *commutative* operators. More precisely, we obtain quantitative version of Theorem 1.2 for the case $l = 2$.

2. On numerical recurrence.

Let X be a metric space with metric $d(\cdot, \cdot)$ and a Borel sigma-algebra of measurable sets Φ . Let T be a measure preserving transformation of a measure space (X, Φ, μ) and let us assume that measure of X is equal to 1. The well-known Poincare (see [9]) theorem asserts that for almost every point $x \in X$:

$$\forall \varepsilon > 0 \forall K > 0 \exists t > K : d(T^t x, x) < \varepsilon.$$

Consider a measure $H_h(\cdot)$ on X , defined as

$$H_h(E) = \lim_{\delta \rightarrow 0} H_h^\delta(E),$$

where $h(t)$ is a positive ($h(0) = 0$) continuous increasing function and $H_h^\delta(E) = \inf_\tau \{\sum h(\delta_j)\}$, when τ runs through all countable coverings E by open sets $\{B_j\}$, $\text{diam}(B_i) = \delta_j < \delta$.

If $h(t) = t^\alpha$ then we get the ordinary Hausdorff measure $H_\alpha(\cdot)$.

We shall say that a measure μ is congruent to a measure H_h , if any μ -measurable set is H_h -measurable.

The following theorems 2.1 and 2.4 were proved in [13] (see also [11, 12]).

Theorem 2.1 *Let X be a metric space with $H_h(X) = C < \infty$ and let T be a measure preserving transformation of X . Assume that μ is congruent to H_h .*

Consider the following function: $C(x) = \liminf_{n \rightarrow \infty} \{n \cdot h(d(T^n x, x))\}$.

Then the function $C(x)$ is μ -integrable and for any μ -measurable set A

$$\int_A C(x) d\mu \leq H_h(A).$$

If $H_h(A) = 0$ then $\int_A C(x) d\mu = 0$ with no demand on measures μ and H_h to be congruent.

Now we introduce the following concept (see [14]).

Definition 2.2 Let G be a totally bounded subset of X . By $N_\varepsilon(G, X)$ denote the minimal cardinality of ε -net of G . The number $H_\varepsilon(G, X)$ is called the ε -entropy of G . Put $N_\varepsilon(X) = N_\varepsilon(X, X)$.

If X is totally bounded then for any δ , we have $N_\delta(X) < \infty$ and $\sum h(\delta_j) \leq N_\delta(X)h(\delta)$. Let h be the function from the definition of H_h . If $N_\delta(X) \leq C/h(\delta)$ then $H_h(X) \leq C$.

Definition 2.3 Let N be a natural number. By $C_N(x)$ denote the function $C_N(x) = \min\{d(T^n x, x) \mid 1 \leq n \leq N\}$. The function $C_N(x)$ will be called N -constant of recurrence for point x .

Theorem 2.4 Let X be totally bounded metric space with metric $d(\cdot, \cdot)$ and function $N(x) = N_x(X)$. Let $\text{diam}(X) = 1$ and T be a measure-preserving transformation of X .

Let $A \subseteq X$ be an arbitrary μ -measurable set and let $g(x)$ be a real nondecreasing function bounded on $[0, 1]$ such that for any $t \in (0, 1]$ there exists Stieltjes integral $\int_t^1 N_A(x)dg(x)$, where $N_A(x) = \min(\mu(A), N_x(A, X)/N)$. Then

$$\int_A g(C_N(x))d\mu \leq \inf_t \{g(t)\mu(A) + \int_t^1 N_A(x)dg(x)\}.$$

The following lemma due to Poincare (see [9, 11]).

Lemma 2.5 Let Y be μ -measurable set and $t \geq 1$. Define

$$Y(t) := \{x \in Y \mid T^i x \notin Y \text{ for all natural } i, 1 \leq i \leq t\}.$$

Then $\mu(Y(t)) \leq 1/t$.

This lemma is the main tool of the prove of Theorems 2.1, 2.4.

Let us now consider the case of two commutative operators. Let S and R be two commutative measure-preserving transformation of X . The next result is the main one of this section.

Theorem 2.6 Let X be a metric space with $H_h(X) = C < \infty$ and let S, R be two commutative measure-preserving transformation of X . Assume that μ is congruent to H_h .

Let us consider the function

$$C_{S,R}(x) = \liminf_{n \rightarrow \infty} \{L^{-1}(n) \cdot \max\{h(d(S^n x, x)), h(d(R^n x, x))\}\},$$

where $L^{-1}(n) = 1/L(n)$.

Then the function $C_{S,R}(x)$ is μ -integrable and for any μ -measurable set A

$$\int_A C_{S,R}(x)d\mu \leq H_h(A).$$

If $H_h(A) = 0$ then $\int_A C_{S,R}(x) d\mu = 0$ with no demand on measures μ and H_h to be congruent.

The next definition is analog of Definition 2.3.

Definition 2.7 Let N be a natural number. By $C_N^{S,R}(x)$ denote the function $C_N^{S,R}(x) = \min\{ \max\{d(S^n x, x), d(R^n x, x)\} \mid 1 \leq n \leq N \}$. The function $C_N^{S,R}(x)$ will be called N -constant of simultaneously recurrence for point x .

Theorem 2.8 Let X be a totally bounded metric space with metric $d(\cdot, \cdot)$ and function $N(x) = N_x(X)$. Let $\text{diam}(X) = 1$ and let S, R be two measure-preserving transformation of X .

Let $A \subseteq X$ be an arbitrary μ -measurable set and let $g(x)$ be a real nondecreasing function bounded on $[0, 1]$ such that for any $t \in (0, 1]$ there exists Stieltjes integral $\int_t^1 N_A(x) dg(x)$, where $N_A(x) = \min(\mu(A), N_x(A, X)L(N))$. Then

$$\int_A g(C_N^{S,R}(x)) d\mu \leq \inf_t \{g(t)\mu(A) + \int_t^1 N_A(x) dg(x)\}.$$

To prove Theorems 2.6 and 2.8, we need several lemmas.

Lemma 2.9 Let $\mathbf{M} = \{M_1, \dots, M_n\}$ be an arbitrary family of μ -measurable sets. Let us assume that for any $x \in X$ there exist at most l sets of the family \mathbf{M} contain x . Then

$$\mu\left(\bigcup_{i=1}^n M_i\right) \geq \frac{1}{l} \sum_{i=1}^n \mu M_i.$$

Proof. The proof is trivial.

The next lemma is the main of this section. Using this lemma we obtain Theorems 2.6 and 2.8 by the same argument as Lemma 2.5 implies Theorems 2.1 and 2.4 (for details see [13]).

Lemma 2.10 Let Y be a μ -measurable set, $t \geq 1$. Define

$$Y(t) := \{x \in Y \mid S^i x \notin Y \text{ or } R^i x \notin Y \text{ for all natural } i, 1 \leq i \leq t\}.$$

Then $\mu(Y(t)) \leq L(t)$.

Proof. Let $t \geq 1$. We may assume for convenience that t to be natural. Define

$$M_{k_1, k_2} = S^{-k_1} R^{-k_2}(Y(t)), \quad 1 \leq k_1, k_2 \leq t.$$

Let $x \in X$. By $A(x)$ denote the set of indexes (k_1, k_2) such that $x \in M_{k_1, k_2}$. Then $A(x) \subseteq [1, t]^2$. If $|A(x)| > t^2 L(t)$, then using Theorem 1.4, we obtain that $A(x)$ contains a corner. Hence there exist $u_1, u_2, u_3 \in Y(t)$ and natural numbers k, m, d such that $x = S^{-k} R^{-m} u_1 = S^{-k-d} R^{-m} u_2 = S^{-k} R^{-k-d} u_3$.

Since S and R are commutative, it follows that $u_1 = S^{-d}u_2 = R^{-d}u_3$. Hence for $u_1 \in Y(t)$ we have $S^d u_1 \in Y(t)$ and $R^d u_1 \in Y(t)$, $d \leq t$. This contradicts the definition of the set $Y(t)$. Hence there exist at most $t^2 L(t)$ the sets M_{k_1, k_2} such that $x \in M_{k_1, k_2}$. Using Lemma 2.9, we get

$$1 \geq \mu\left(\bigcup_{k_1, k_2} M_{k_1, k_2}\right) \geq \frac{1}{t^2 L(t)} \sum_{k_1, k_2} \mu(M_{k_1, k_2}) = \frac{\mu(Y(t))}{L(t)} \quad (4)$$

Whence $\mu Y(t) \leq L(t)$ as required.

Now we apply Theorem 2.6 to the case of compact metric space.

The following lemma can be found in [15].

Lemma 2.11 *Let X be a compact metric space and let T_1, \dots, T_l be continuous commutative transformations of X . Then there exists a finite measure μ such that transformations T_1, \dots, T_l preserve μ .*

Corollary 2.12 *Let X be a compact metric space with metric $d(\cdot, \cdot)$ and $H_h(X) < \infty$. Let S, R be two continuous commutative transformations of X . Then there exists $x \in X$ such that*

$$\liminf_{n \rightarrow \infty} \{L^{-1}(n) \cdot \max\{h(d(S^n x, x)), h(d(R^n x, x))\}\} \leq C.$$

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